angular velocity between the propellers. Ω_{cp} is the radial distribution of the angular velocities of nested fans that would produce the same wake as the coaxial pair of propellers. For a swirlless counterrotating pair, $\omega=0$, $\Omega_{cp}=\infty$. If Ω_I or $\Omega_{II}=0$, the front or rear propeller becomes a set of fixed vanes that can pre- or deswirl the flow.

Unlike Eq. (1), Eq. (8) is not entirely in terms of wake quantities. The term $\omega_b r_b^2$, which is the angular momentum of a unit volume of fluid between the propellers at radius r_b, on the same streamline as r in the wake, cannot be inferred from the flow in the wake. The second propeller makes its own arbitrary contribution to the swirl, which renders wake swirl useless as evidence of the swirl between the propellers.

Conclusions

If the longitudinal component of velocity is uniform across a propeller wake, the swirl energy in the wake is exactly paid for by the work done by wake suction. For lightly loaded propellers the longitudinal velocity is always fairly uniform, and the relation between swirl energy and wake suction is a good approximation. Either swirl energy or wake suction may be used to measure swirl loss, but not both at once. It is not their sum.

Appendix

Let the propellers be close enough together so one can ignore the convergence of the flow between them and take the longitudinal component of velocity on any streamline as constant from one propeller to the next. The swirl component and the pressure can change at each propeller. Swirl angular velocity between the propellers is ω_b, that immediately after the second propeller is ω_a (the subscripts standing for between and after). The pressure jump across the upstream propeller is p'_I, across the downstream propeller p'_{II}, across both propellers $p'=p'_I+p'_{II}$.

Applying Bernoulli's equation to the velocity of the fluid relative to a point fixed to the upstream propeller at radius r_b, and rotating with it, relates the pressure increase across the propeller disk to the angular velocity of the propeller and the swirl downstream of it, thus

$$p'_I=(\rho/2)[\Omega_I^2-(\Omega_I-\omega_b)^2]r_b^2$$

$$=\rho(\Omega_I-\omega_b/2)\omega_b r_b^2 \tag{A1}$$

Because the longitudinal velocity is the same up and downstream of the propeller, it does not appear in the equation.

The swirl component of the velocity of the fluid entering the second propeller relative to that of the propeller is $(\omega_b-\Omega_{II})r_b$, and of that leaving it is $(\omega_a-\Omega_{II})r_b$. By the same Bernoulli argument, the pressure rise across the downstream propeller is

$$p'_{II}=(\rho/2)[(\Omega_{II}-\omega_b)^2-(\Omega_{II}-\omega_a)^2]r_b^2 \tag{A2}$$

so

$$p'=p'_I+p'_{II}=\rho[(\Omega_I-\Omega_{II})w_b+\Omega_{II}w_a-\omega_a^2/2]r_b^2 \tag{A3}$$

Applying Bernoulli's theorem separately to the flow that accelerates from infinity to the upstream face of a propeller, and to the flow that accelerates from the downstream face to infinity gives

$$p=(\rho/2)(V^2-u^2+\omega_a^2 r_b^2-\omega^2 r^2)+p' \tag{A4}$$

Combining this equation with Eq. (A3) gives

$$p=(\rho/2)(V^2-u^2)+\rho\{[(\Omega_I-\Omega_{II})\omega_b+\Omega_{II}\omega_a]r_b^2-\omega r^2/2\} \tag{A5}$$

which reduces to Eq. (1) if either propeller is missing, that is, if $\omega_b=0$ or $\omega_b=\omega_a$.

Using the conservation of angular momentum relation

$$\omega_a r_b^2=\omega r^2 \tag{A6}$$

the equation can be transformed into

$$p=(\rho/2)(V^2-u^2)+\rho[(\Omega_I-\Omega_{II})\omega_b r_b^2+(\Omega_{II}-\omega/2)\omega r^2] \tag{A7}$$

which can be reduced to Eq. (1) by the substitution

$$\Omega_{cp}=(\Omega_I-\Omega_{II})(\omega_b r_b^2)/(\omega r^2)+\Omega_{II} \tag{A8}$$

References

[1]Gearhart, W. S., "Proposed Arrangement to Improve Turboprop Efficiency," *Journal of Aircraft*, Vol. 21, May 1984, pp. 341-345.

[2]Glauert, H., "Airplane Propellers," *Aerodynamic Theory*, edited by W. F., Durand, Vol. 4, Dover, New York, 1963, pp. 191-192.

The Unsuitability of Ellipsoids as Test Cases for Line-Source Methods

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RECENTLY interest has reawakened in calculating axisymmetric potential flow about bodies of revolution by means of line-source distributions on the symmetry axis.[1,2] This technique was first put forward by von Kármán.[3] The general procedure is to divide a portion of the x-axis inside the body into a number of line-source elements, over each of which the source density has a known variation: constant, linear, quadratic, etc. The combined stream function of the axial uniform stream plus the line-source distribution is set equal to zero at a number of points on the body profile, which yields a set of linear equations for the parameters defining the axial source distribution.

There are two problem areas for such methods. First, as von Kármán warned,[3] not every body can be represented by an axial source distribution. Second, the line source cannot extend to the ends of the body at finite strength or the velocities there will be infinite. Thus, the ends of the source distribution must be inset some distance from the ends of the body, and determination of this inset distance is both important and nonstraightforward. To address these problems and to demonstrate the accuracy of their methods, investigators com-

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pare results calculated by their methods with analytic solutions.

Because of the simplicity of its analytic solution, the case most often selected for comparison is the prolate spheroid of various thickness ratios. However, it is a classic result of potential theory that a prolate spheriod can be represented exactly by a linearly varying line source lying between the foci (see below). Thus, the prolate spheriod is totally unsuitable as a test case for line-source methods. If the source distribution lies between the foci and a linear or higher degree variation is used on the line elements, the calculational error should be exactly zero for any number of elements down to and including a single one. If the line elements are constant sources, the results are not exact, but clearly this is a very favorable nonrepresentative case for a line-source method.

Zedan and Dalton[1] use linear and quadratic line elements distributed between the foci of an ellipsoid with thickness ratio of 0.125. Calculational errors are very low but not exactly zero, presumably because of roundoff error and an approximate determination of the foci. Kuhlman and Shu[2] perform numerical experiments to determine the optimum inset distance for ellipsoids with thickness ratios of 0.1, 0.2, and 0.5. Their results represent the focus location to the decimal places given for the first two but are 10% low for the third. They also present a formula for inset distance as a function of thickness ratio, which is the correct limiting form for small values of the latter. Finally, Kuhlman and Shu[2] present calculational error vs element number for constant and linear source and doublet elements. The linearly varying source elements give the lowest error, but it is not exactly zero, probably for the reasons above. All these experiments are manifestations of the classic analytic result.

The fact that the axisymmetric flow about a prolate spheroid can be obtained by superposing a uniform stream and the flow due to a linearly varying line source between the foci was apparently once well known but has been largely forgotten. Accordingly, proofs are hard to find. It is stated without proof by Birkhoff[4] and Moran,[5] the latter of which also presents a thorough investigation of inset distance. Munk[6] presents a proof from a particular viewpoint. Fortunately this result is easy to derive from first principles by equating to zero the combined stream function of the uniform stream and the line source—a very nice classroom demonstration. Finally, the prolate spheroid provides a counterexample to the frequently repeated claim that axial source distributions can represent only thin bodies, since prolate spheriods can be so represented exactly up to a thickness ratio of unity.

References

[1]Zedan, M.F. and Dalton, C., "Higher-Order Axial Singularity Distributions for Potential Flow About Bodies of Revolution," *Computer Methods in Applied Mechanics and Engineering,* Vol. 21, 1980, pp. 295-314.

[2]Kuhlman, J.M. and Shu, J.-Y., "Potential Flow Past Arbitrary Bodies at Angle of Attack," *Journal of Aircraft,* Vol. 21, March 1984, pp. 218-220.

[3]von Kármán, T., "Calculation of Pressure Distributions on Airship Hulls," NACA TM 574, 1930.

[4]Birkhoff, G., *Hydrodynamics,* Princeton University Press, Princeton, N.J., 1960, p. 155.

[5]Moran, J.P., "Line Source Distributions and Slender-Body Theory," *Journal of Fluid Mechanics,* Vol. 17, Part 2, 1963, pp. 285-304.

[6]Munk, M., *Fluid Dynamics, Part II. Aerodynamic Theory,* edited by W.F. Durand, Julius Springer, Berlin, 1935, p. 284.